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Minimal cones and Plateau's problem again?

(Mostly, old results and questions)

Princeton, May 2011

Guy David, Université de Paris Sud 11 (Orsay)

[Happy birthday Elias!]

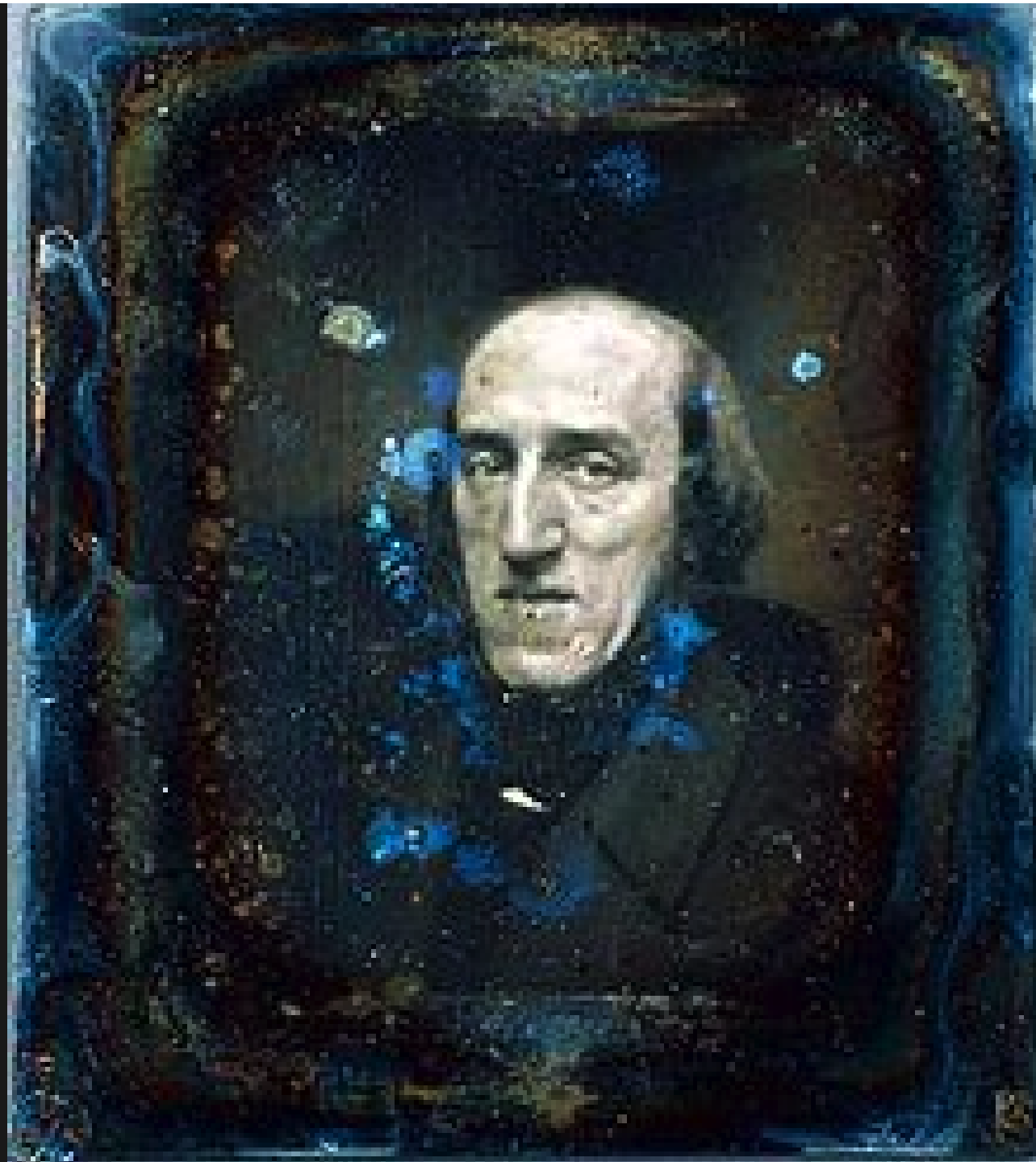
1. THE PLATEAU PROBLEM (some attempts)

Plateau was interested in soap films. The simplest statement of Plateau's problem: describe the soap films $E \subset \mathbb{R}^3$ bounded by a set Γ (for instance a smooth curve). Existence and regularity?

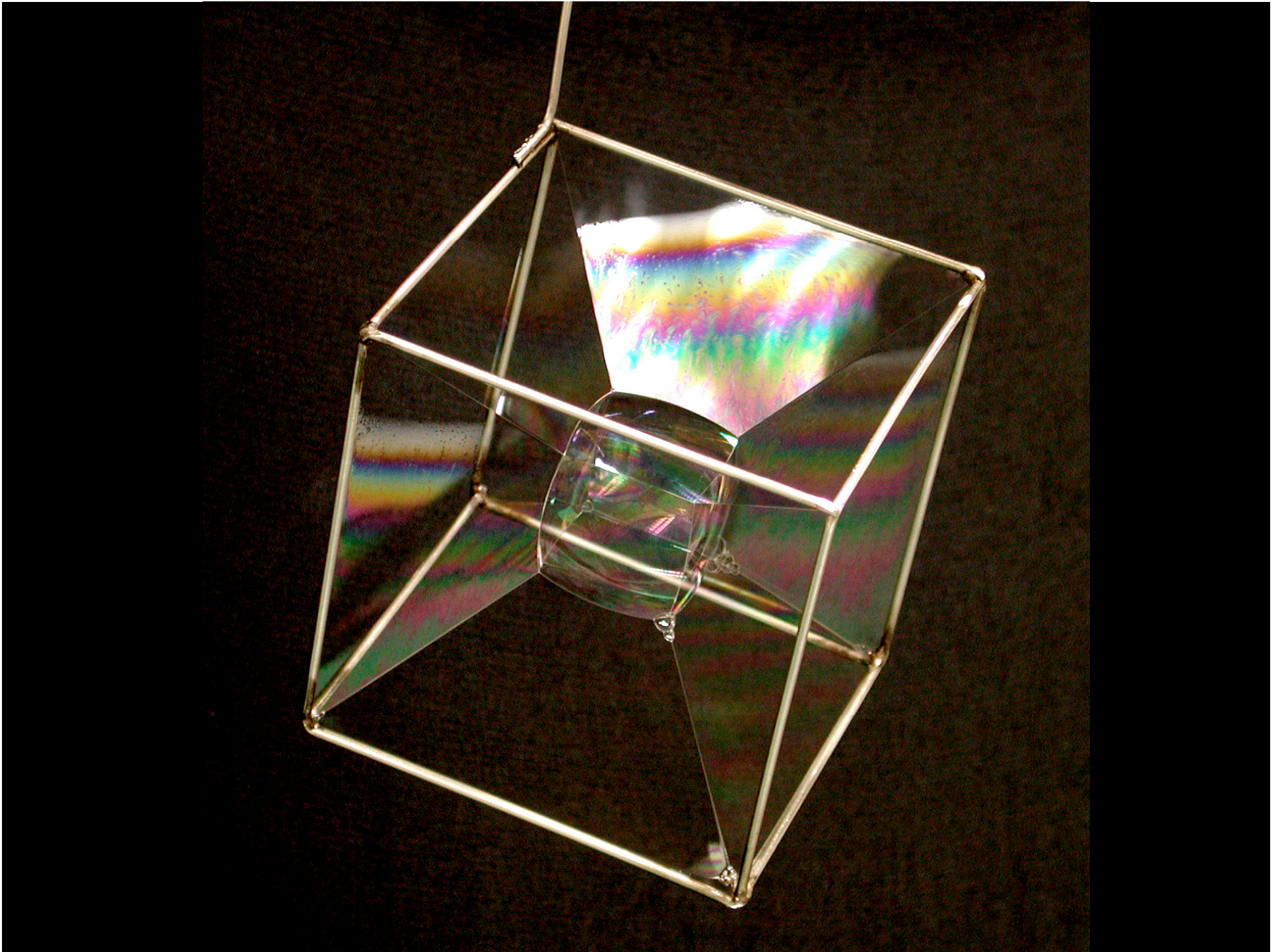
Apparently, a soap film is composed of two layers of some molecules with a water-attracting head and a water-repelling tail, which align themselves head to head; the width of the film is roughly equal to the length of two molecules.

The general model is to minimize the area of a surface E spanned by Γ , but different notions of “surface” and “spanned” exist. We shall mention some.

A soap bubble is slightly different: due to different pressures on both sides, it only “almost-minimizes” the area. In the smooth case, it has constant (instead of vanishing) mean curvature.







1.a. Douglas' solution of Plateau's problem (1931)

Take a smooth curve Γ , and look for $E = f(D)$, where D is the unit disk. Try to minimize the area of the image $\int_D J_f(x)dx$.

Difficulty: too many possible parameterizations to hope for compactness.

Bright idea: try to take f to be harmonic in D , and choose f on ∂D so that $\int_D J_f(x)dx$ is minimal. Luckily, this amounts to solving a variational problem for $f|_{\partial D}$ which is much simpler: minimize

$$A(f) = \int \int \frac{\sum_{j=1}^n |f_j(\theta) - f_j(\varphi)|^2}{\sin^2\left(\frac{\theta-\varphi}{2}\right)} d\theta d\varphi$$

But the method does not see whether $f(D)$ crosses itself.

1.b. Currents and the Plateau problem

We start with the most celebrated and successful model, provided by Currents. Work by Federer, Fleming, De Giorgi, and others.

A d -dimensional **current** is a continuous linear form on the space of smooth d -forms. Almost the same as a form-valued distribution.

Main example: if S is smooth, oriented surface of dimension d , the current S' of integration on S is defined by $\langle S', \omega \rangle = \int_S \omega$. But we want a much larger class with compactness properties.

Other useful example, the rectifiable current T defined on a d -dimensional rectifiable set E such that $\mathcal{H}^d(E) < +\infty$, with a measurable orientation τ , and an integer-valued multiplicity m :

$$(1) \quad \langle T, \omega \rangle = \int_E m(x) \omega(x) \cdot \tau(x) d\mathcal{H}^d(x).$$

Here $\mathcal{H}^d = d$ -dimensional Hausdorff measure \simeq surface measure.

Recall (1) :
$$\langle T, \omega \rangle = \int_E m(x) \omega(x) \cdot \tau(x) d\mathcal{H}^d(x).$$

We omit the definition of $\omega(x) \cdot \tau(x)$ when τ is a measurable orientation (or unit d -vector), but it would be the same on a smooth manifold.

The boundary ∂T of a d -dimensional current T is defined by

(2)
$$\langle \partial T, \omega \rangle = \langle T, d\omega \rangle \text{ for every } (d-1)\text{-form } \omega.$$

[d is the exterior derivative.] When S is a smooth oriented surface with boundary Γ , Green says that $\partial S' = \Gamma'$. Notice: $\partial\partial = 0$ because $dd = 0$.

Boundary condition for currents: take a $(d-1)$ -dimensional current Γ , with $\partial\Gamma = 0$, and minimize “area” among d -dimensional currents T such that $\partial T = \Gamma$.

Works well and great success for GMT : minimize $\text{Mass}(T)$, which is the operator norm of T , where we put a L^∞ -norm on forms.

Federer, De Giorgi, etc., get existence and regularity of normal currents T that solve $\partial T = \Gamma$ and minimize $\text{Mass}(T)$.

Very good, but not very physical (for instance, T has no interior singularity in low dimensions).

More physical option: minimize $\text{Mass}(T)$, the \mathcal{H}^d -measure of the support of T .

But then, no general existence result, even when $n = 3$, $d = 2$, and Γ is a smooth curve. The difficulty is that we have no bounds on the masses in a minimizing sequence, so the good compactness theorem does not apply.

Interesting partial results by R. Hardt and T. De Pauw.

Difference : when T is a rectifiable current given by

$$(1) \quad \langle T, \omega \rangle = \int_E m(x) \omega(x) \cdot \tau(x) d\mathcal{H}^d(x),$$

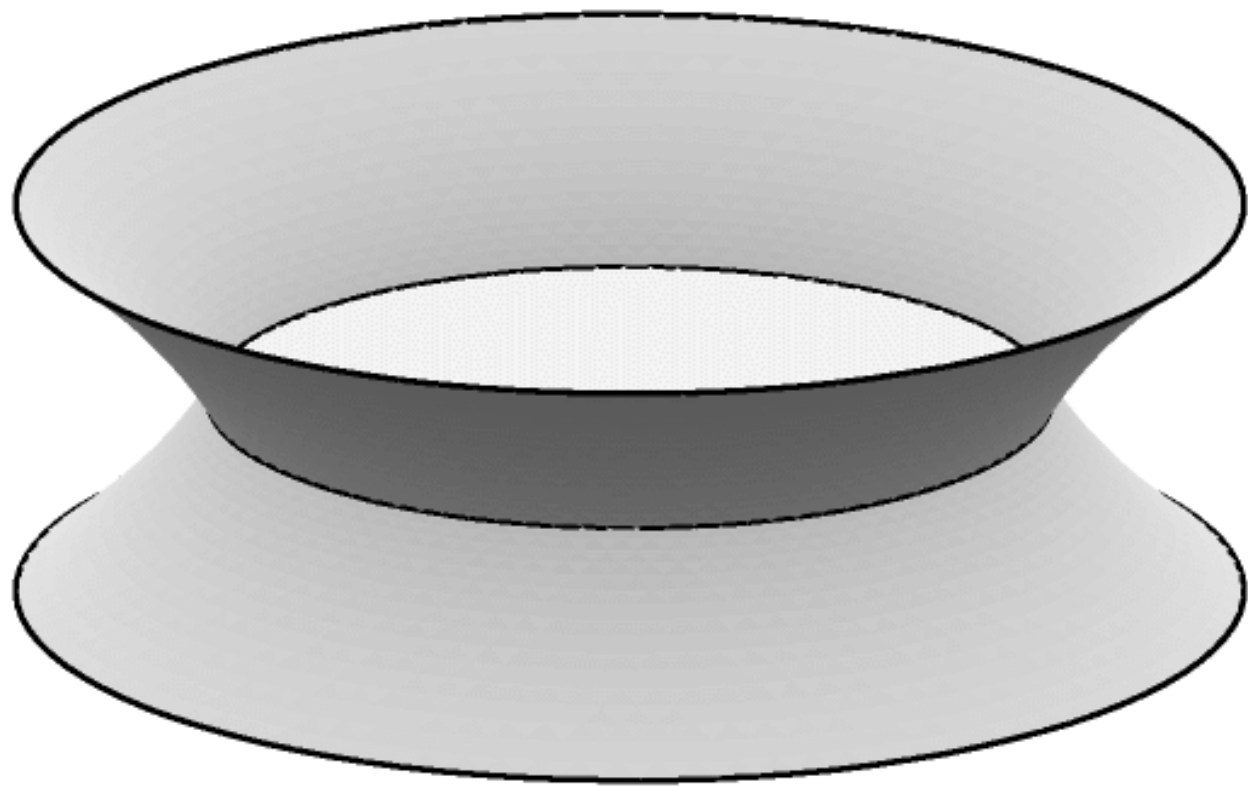
$$(3) \quad \text{Mass}(T) = \int_E |m(x)| d\mathcal{H}^d(x)$$

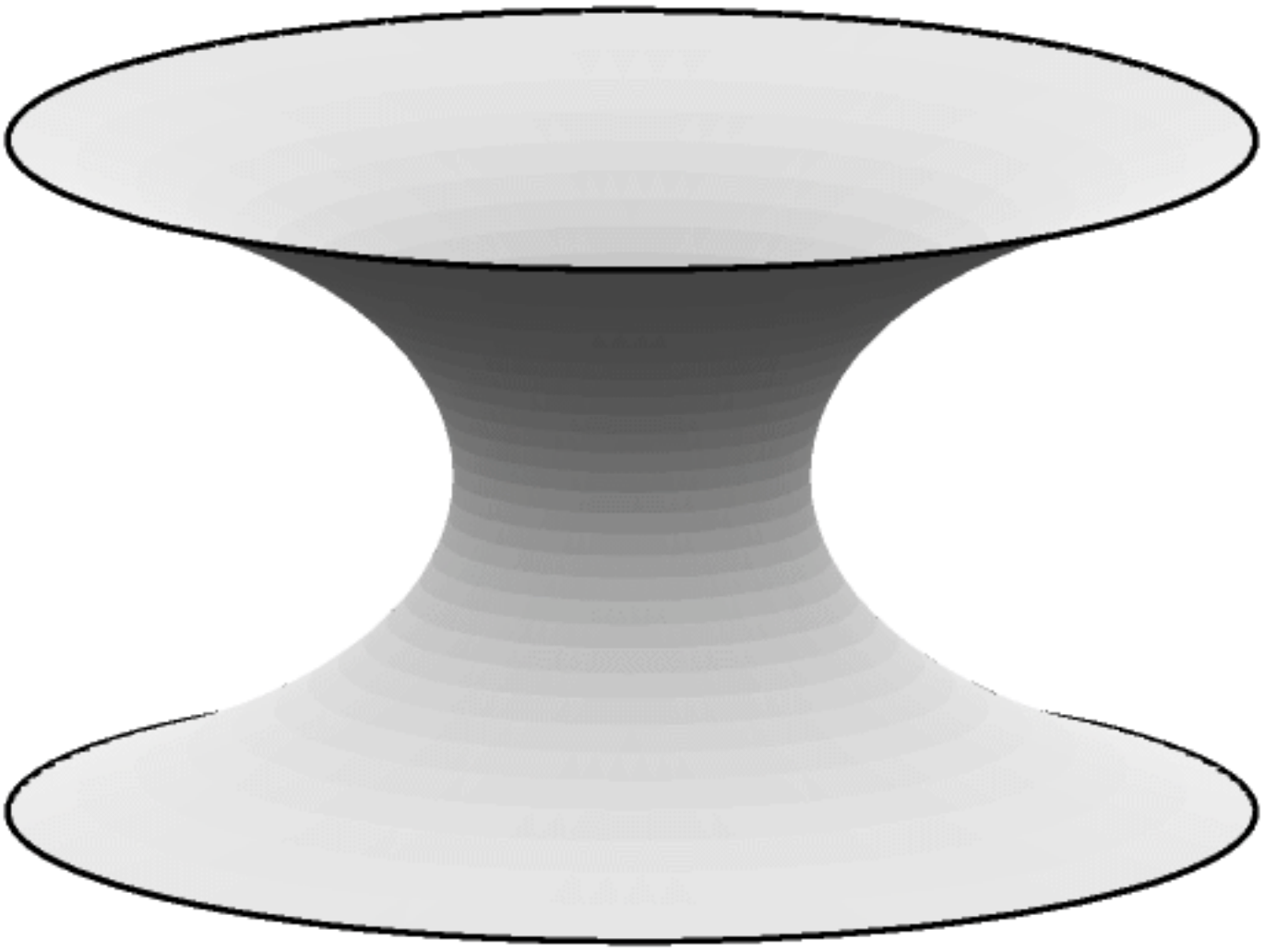
and

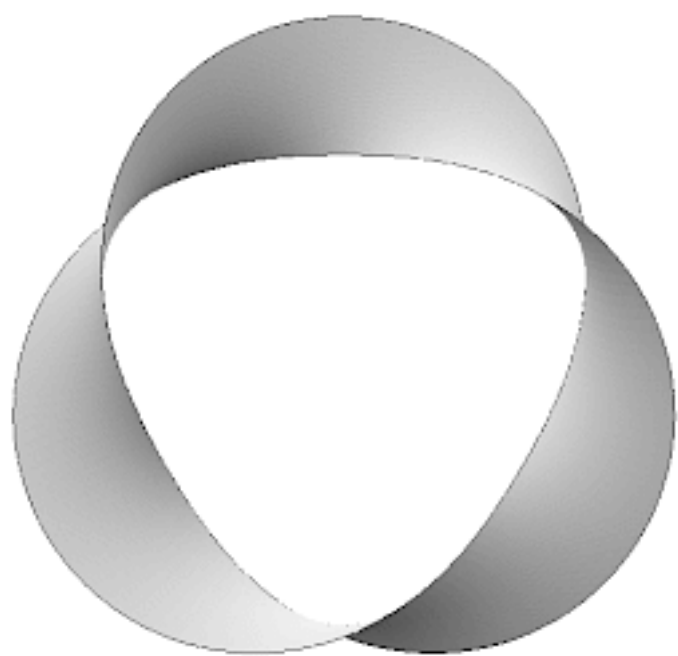
$$(4) \quad \text{Size}(T) = \mathcal{H}^d(\{x \in E; m(x) \neq 0\})$$

Simple pictures when $d = 1$ and $d = 2$.

Definition : A normal current is a rectifiable current T such that ∂T is rectifiable too. [Did not disturb here because we know ∂T .]







1.c. Directly with sets and homology

Another difficulty with currents: some soap films are not orientable (Möbius bands), and the problem $\partial T = \Gamma$ does not always fit.

Various ad hoc solutions to this exist, but no general scheme.

Return to $d = 2$, $n = 3$, and E is a “surface” spanned by a curve Γ . We want to minimize $\mathcal{H}^2(E)$, but what is “spanned”?

For Reifenberg (1960), E is a compact set that contains Γ , and the boundary condition is in terms of Čech homology on some commutative group G . We require the inclusion $i : \Gamma \rightarrow E$ to induce a trivial homomorphism from $H_1(\Gamma, G)$ to $H_1(E, G)$. Then we minimize $\mathcal{H}^2(E)$.

Reifenberg proved the existence of minimizers when $G = \mathbb{Z}_2$ or \mathbb{R}/\mathbb{Z} . Beautiful proof with minimizing sequences and haircuts.

De Pauw obtained the 2-dimensional case when $G = \mathbb{Z}$ (with currents). The equivalence with the size minimizing problem is not clear, but the infimum is the same.

1.d. Sliding Almgren minimizers

We propose a third definition, where we minimize $\mathcal{H}^2(E)$ among all compact sets E obtained by deformation of an initial candidate E_0 with a sliding boundary condition. [Think about a rubber shower curtain.]

A deformation of E_0 with sliding boundary condition is a set $E = \varphi_1(E_0)$, where $\varphi_t : E_0 \rightarrow \mathbb{R}^3$, $0 \leq t \leq 1$, is a one-parameter family of functions such that:

$$(7) \quad (x, t) \rightarrow \varphi_t(x) \text{ is continuous: } E_0 \times [0, 1] \rightarrow \mathbb{R}^3,$$

$$(8) \quad \varphi_0(x) = x \text{ for } x \in E_0,$$

$$(9) \quad \varphi_t(x) \in \Gamma \text{ when } x \in \Gamma,$$

$$(10) \quad \varphi_1 \text{ is Lipschitz.}$$

[We require (10) for safety, but no bound on the Lipschitz constant is attached. We may also drop (10).]

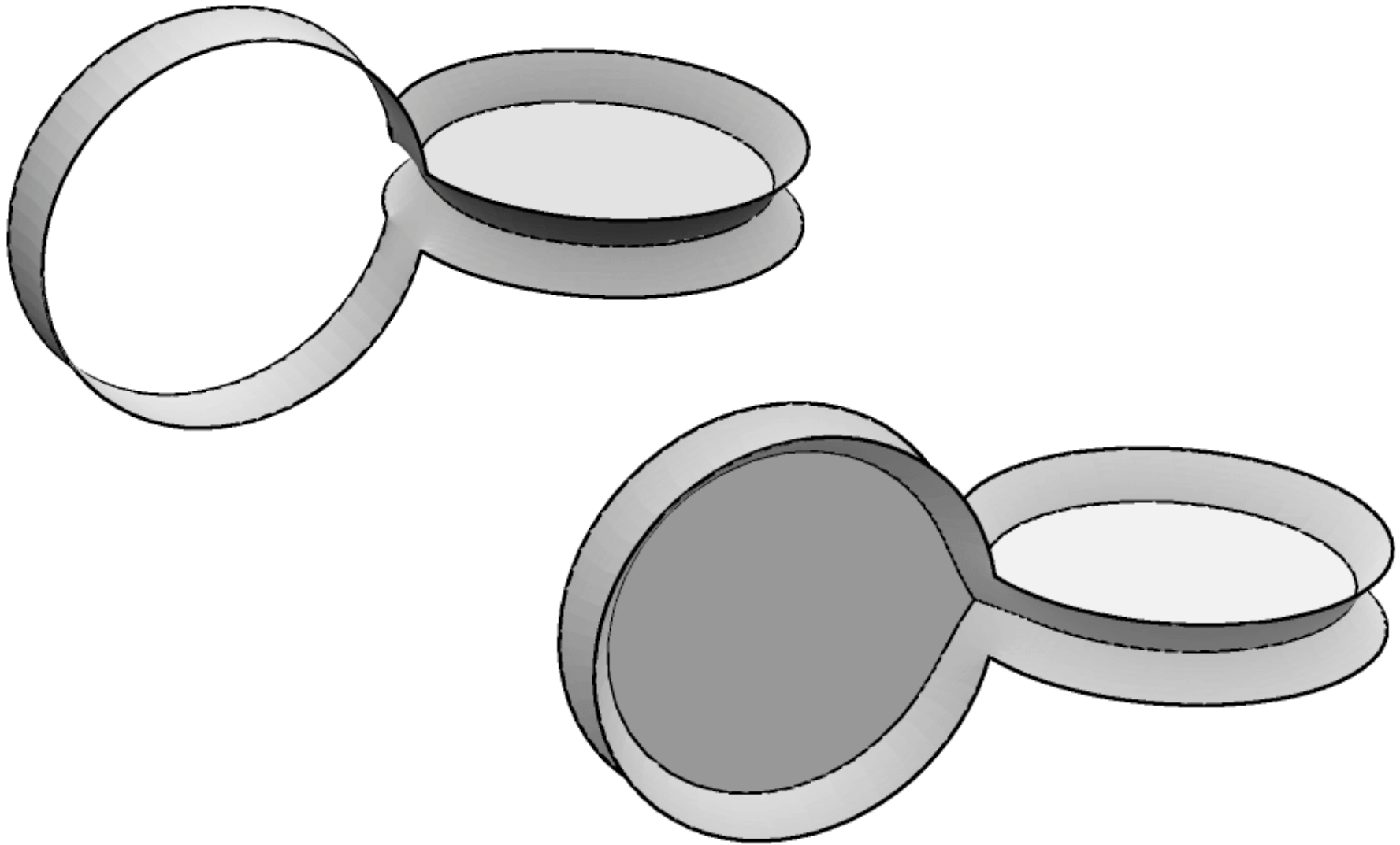
Possible advantages of this notion:

- No need to orient E , or choose a group.
- Could be more flexible (different choices of E_0 would lead to different solutions).

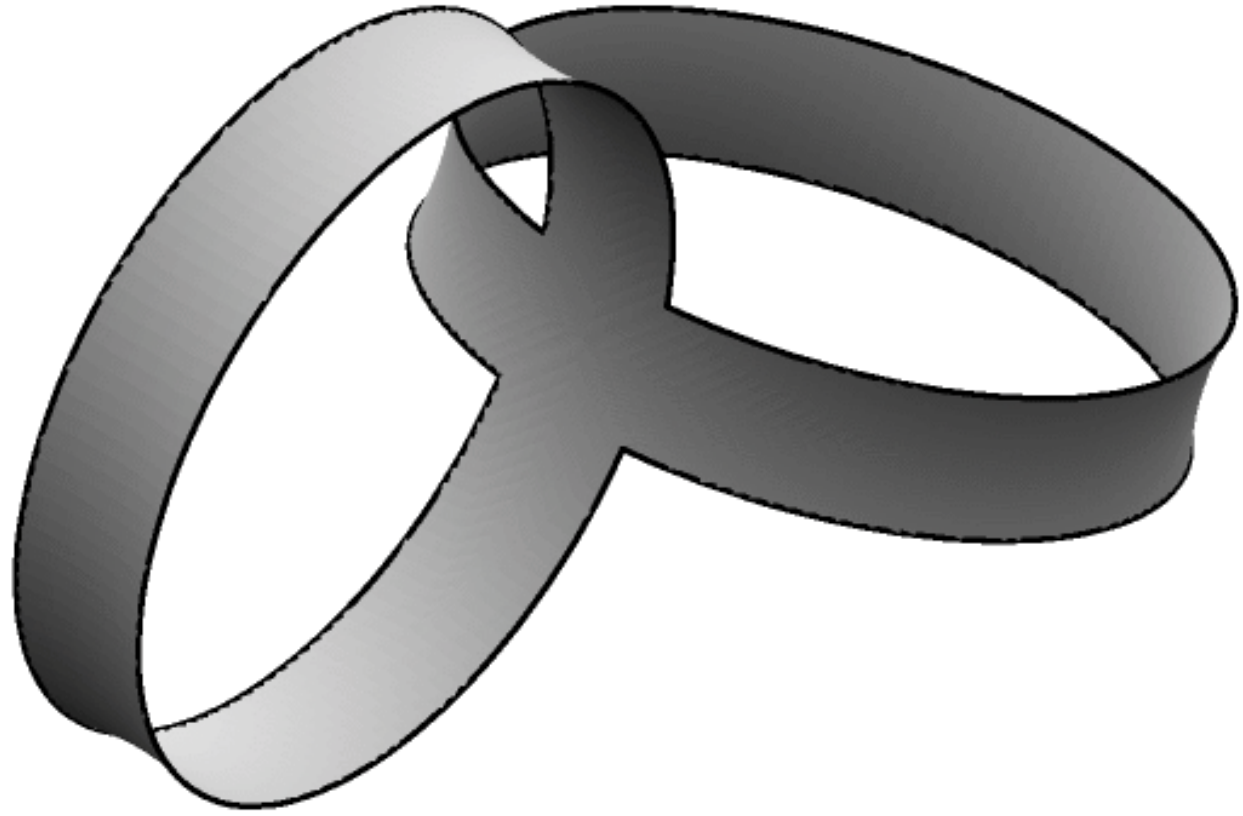
Bad news: no existence result is known yet even when $d = 2$, $n = 3$, and the boundary Γ is a curve. Also, we do not account for unrealistic deformations that would extend the film too far: some real films could be deformed into a point, but with a long homotopy.

All these definitions work in a much wider context (but then even less is known). Other ones are possible. Separation conditions, constraints on the homology of the complement are easier.

We leave the Plateau problem with the conclusion that it would be nice to solve it in simple various context.



Pictures from K. Brakke's site



2. INTERIOR REGULARITY : J. TAYLOR'S THEOREM

We now focus on regularity properties of potential solutions away from the boundary. J. Taylor's theorem will be the best example.

First we'll define local minimal and almost-minimal sets.

We shall use Almgren's definition, which gives a good description of soap films and bubbles, of a d -dimensional almost-minimal set.

We localize to an open set $U \subset \mathbb{R}^n$. For instance, U could be $\mathbb{R}^n \setminus \Gamma$, where Γ is a boundary.

We use a small gauge function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, that measures how close to minimal. For instance, $h(r) = Cr^\alpha$ for some $\alpha > 0$; take $h = 0$ for locally minimal sets.

We consider sets $E \subset U$, of dimension d , and even such that $\mathcal{H}^d(E \cap B) < +\infty$ for every compact ball $B \subset U$.

Let B be a compact ball in U . A competitor for E in B is a set $F = \varphi(E)$, where $\varphi : U \rightarrow U$ is Lipschitz (but no bounds required), and

$$(1) \quad \varphi(y) = y \text{ for } y \in U \setminus B, \text{ and } \varphi(B) \subset B.$$

An almost minimal set of dimension d in U , with gauge function h , is a closed set E , with $\mathcal{H}^d(E \cap B) < +\infty$ for closed balls $B \subset U$, such that

$$(2) \quad \mathcal{H}^d(E \cap B) \leq \mathcal{H}^d(F \cap B) + r^d h(r)$$

whenever B is a closed ball, $B \subset U$, r is its radius, and F is a competitor for E in B .

- Comments. 1. Minor variant of Almgren's definition. Others exist.
 2. F is obtained by deforming E inside B , because $F = \varphi_1(E)$, where $\varphi_t(y) = t\varphi(y) + (1-t)y$.
 3. Note that φ_1 is **not** required to be injective. Pinching is allowed.

Examples of Minimal cones

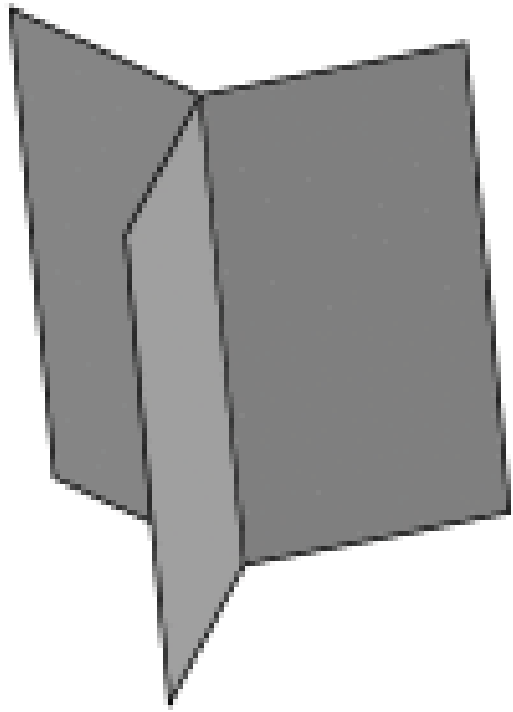
That is, cones that are also minimal sets, i.e., such that

$$\mathcal{H}^d(E \cap B) \leq \mathcal{H}^d(\varphi(E) \cap B)$$

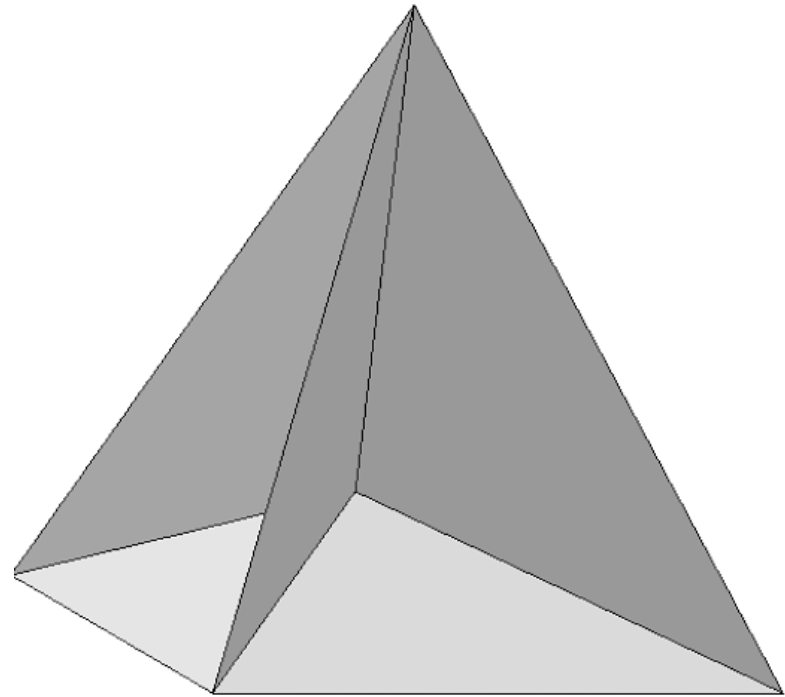
for $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ Lipschitz s.t. $\varphi(x) = x$ for $x \in \mathbb{R}^n \setminus B$ and $\varphi(B) \subset B$.

- $d = 1$: A line, a Y (three half lines in a plane, with 120° angles).
- $d = 1$: But not two lines (even perpendicular). Exercise: the union of two crossing segments is not (the support of) a mass or size minimizer.
- $d = 2$: A plane, a \mathbb{Y} (product of Y by a perpendicular line = three half planes that make 120° angles).
- $d = 2$: Less trivial: in \mathbb{R}^3 , the (closed positive) cone \mathbb{T} of dimension 2 over the union of the edges of a regular tetrahedron centered at the origin. Six faces that meet by sets of three along four half lines (the spine). [Morgan-Lawlor with calibrations.]

Full list in \mathbb{R}^3 , known by Lamarle, Heppes, Taylor.



Y



T

Pictures from Brakke's site

Reduction (= cleaning)

For $E \subset \mathbb{R}^n$ closed, with locally finite \mathcal{H}^d measure, denote by E^* the closed support of E . That is,

$$(3) \quad E^* = \{x \in E; \mathcal{H}^d(E \cap B(x, r)) > 0 \text{ for all } r > 0\}.$$

Say that E is reduced when $E = E^*$.

If E is almost minimal, then E^* is almost minimal, with the same gauge h , because $\mathcal{H}^d(E \setminus E^*) = 0$. So it is safe to focus on reduced sets.

This simplifies things; otherwise we would get ugly statements because if E is almost minimal, then $E \cup Z$ is also almost minimal for any closed Z such that $\mathcal{H}^d(Z) = 0$.

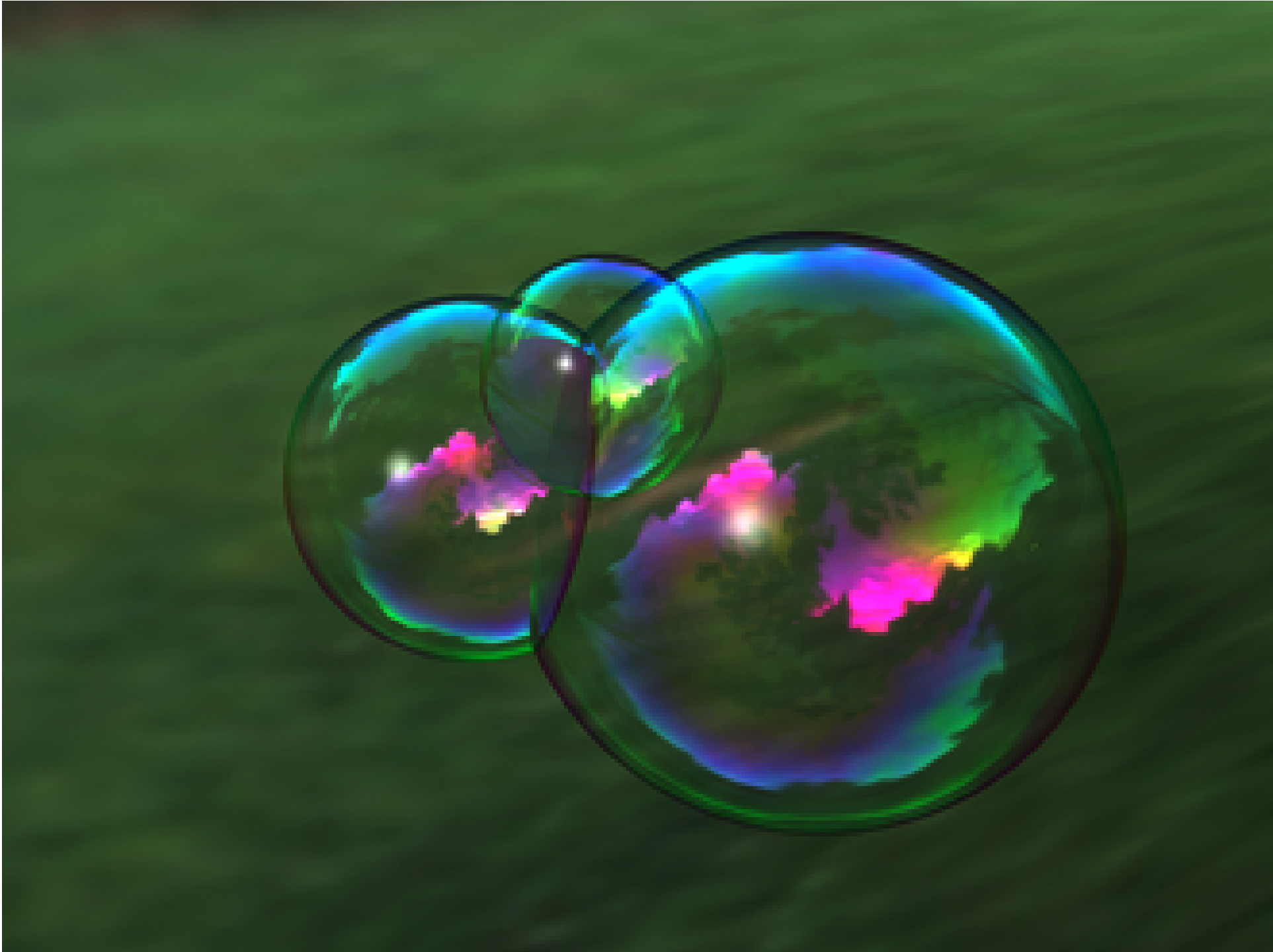
We should keep in mind that $E^* \setminus E$ can play a role in some topological problems. But from now on, all our sets will be reduced.

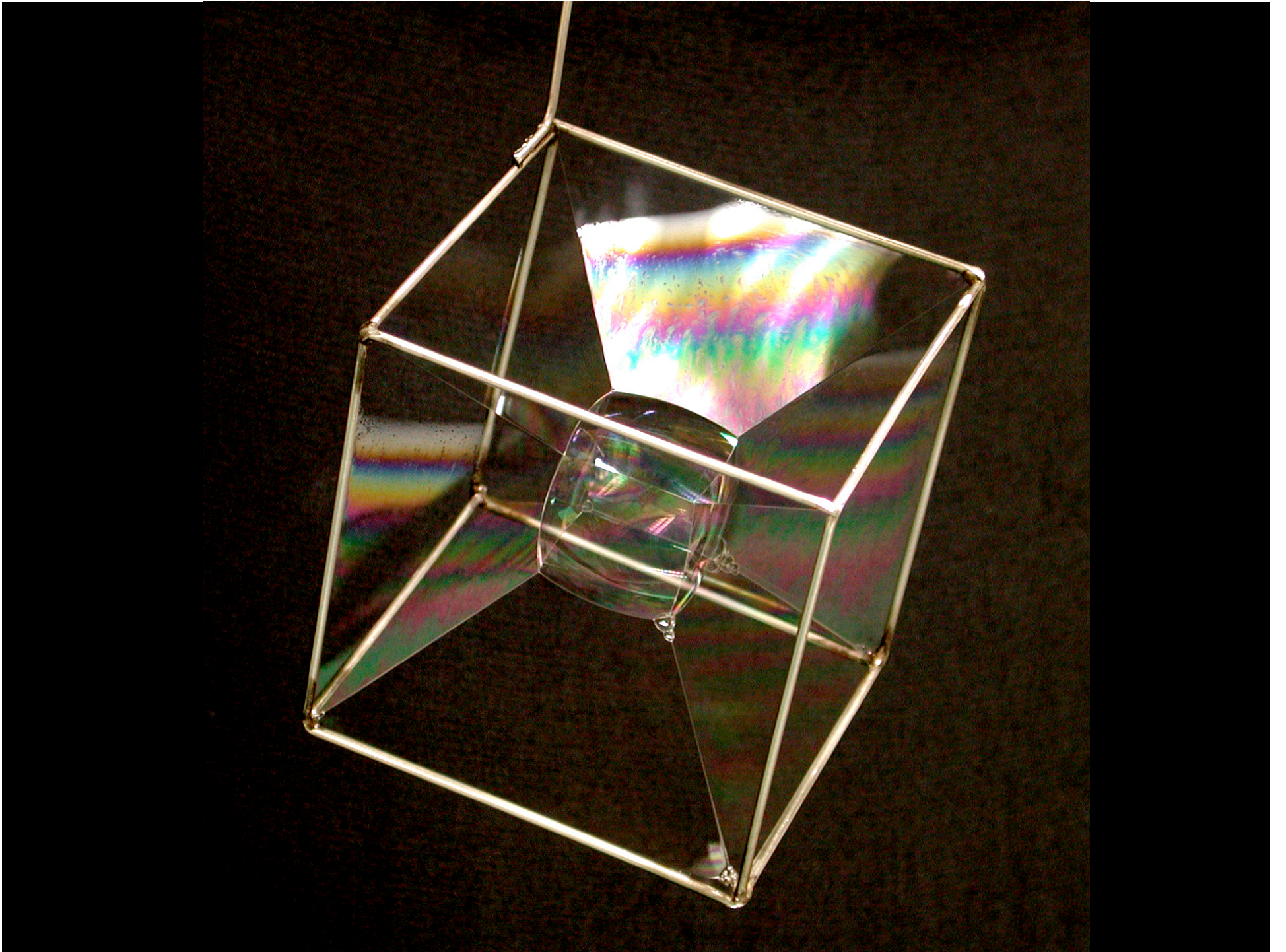
J. Taylor's theorem

Theorem (JT, 1976). Let E be a reduced local almost minimal set of dimension 2 in some open set $U \subset \mathbb{R}^3$, with gauge function $h(r) = Cr^\alpha$ ($\alpha > 0$). Then for each $x \in E$, there is a ball $B(x, r)$ inside which E is the image of a minimal cone by a C^1 -diffeomorphism of \mathbb{R}^3 .

Comments

- The cone is a plane, a \mathbb{Y} , or a \mathbb{T} , as in the list above. This also means that near $B(x, r)$, E is composed of C^1 faces, which meet along C^1 arcs with 120° angles and the same combinatorics as in a minimal cone.
- The singularities above occur in real soap films.
- Alas, the result does not give a concrete way to estimate r .
- Some (Dini) condition on h is needed, but $h(r) = Cr^\alpha$ is not optimal.
- You can get more regularity than this, especially if h is very small or vanishing and in the regions where E is C^1 -diffeomorphic to a plane.





Some ingredients for the proof.

- (Almost) monotonicity of the density $\theta_x(r) = r^{-d}\mathcal{H}^d(E \cap B(x, r))$ (standard for minimal sets, obtained by comparing with a cone).
- Stability under limits (which comes from other regularity properties). And then the blow-up limits of E are minimal cones.
- More precise arguments, to show that at small scales, E stay close to minimal cones, with uniform estimates. Includes a topological lemma of existence of points of type Y .
- A variant of Reifenberg's theorem on parameterizations, which gives the local Bi-Hölder equivalence.
- For the C^1 estimates, much more precise comparisons with minimal cones; epiperimetry.

Knowing the list of all minimal cones is a very important first step anyway (because of the blow-up limits).

3. MINIMAL CONES AND SETS IN HIGHER DIMENSIONS

Here this will just mean, an analogue of J. Taylor's theorem for 2-dimensional minimal sets in \mathbb{R}^n :

Theorem [D.]. Let E be a reduced local almost minimal set of dimension 2 in some open set $U \subset \mathbb{R}^n$, with gauge function $h(r) = Cr^\alpha$ ($\alpha > 0$). Then for each $x \in E$, there is a ball $B(x, r)$ where E is the image of a minimal cone by a bi-Hölder diffeomorphism of \mathbb{R}^n .

Good, but what is the list of minimal cones?

This would also help for proving the C^1 -equivalence.

We have the following first description of 2-dimensional minimal cones E in \mathbb{R}^n . Set $K = E \cap \partial B(0, 1)$. Then K is a finite union of circles and arcs of great circles; the arcs are not too short and meet at their ends by sets of 3 and with 120° angles; the circles stay at some positive distance from the rest. [First proof by F. Morgan.]

Examples :

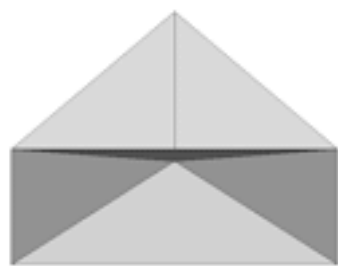
In \mathbb{R}^3 , \mathbb{P} comes from a circle; \mathbb{Y} from 3 half circles that meet at the poles, \mathbb{T} from 6 shorter arcs. The other nets on \mathbb{S}^2 do not give minimal cones. See pictures from Ken Brakke's home

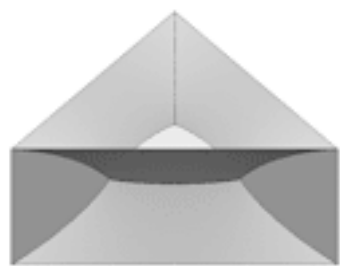
<http://www.susqu.edu/brakke/>

In \mathbb{R}^4 , two new potential examples :

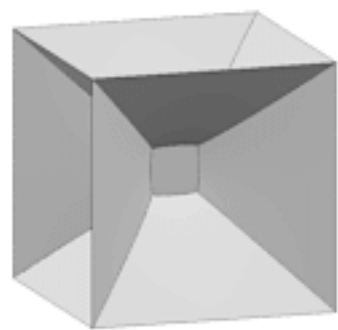
- The union $P_1 \cup P_2$ of two transverse 2-planes. Known to be minimal when they are almost orthogonal [XiangYu Liang]; Conjectural necessary and sufficient condition for minimality [Morgan].
- Product $Y \times Y$ of two one-dimensional sets Y (9 faces). Not even known to be minimal.
- Is there any other one?

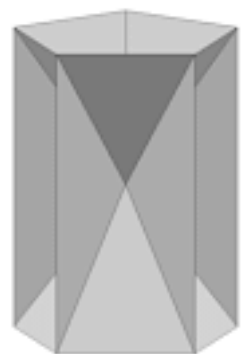
For 3-dimensional minimal sets and cones in \mathbb{R}^4 , little is known. Smooth description of $K = E \cap \partial B(0, 1)$ when E is a minimal cone [Luu].

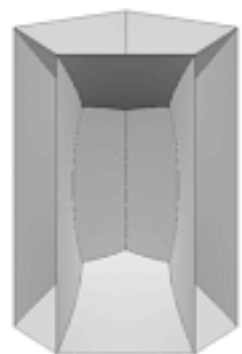












4. BOUNDARY REGULARITY?

Idea: control the behaviour of E near boundary pieces; hope this could also lead to existence results for Plateau.

Setting: Now we are given a collection of (closed) boundary pieces Γ_j , $j \in J$. A sliding competitor for E is a set $F = \varphi_1(E)$, where the φ_t , $0 \leq t \leq 1$, are such that

$$(1) \quad (x, t) \rightarrow \varphi_t(x) \text{ is continuous: } E \times [0, 1] \rightarrow \mathbb{R}^3,$$

$$(2) \quad \varphi_0(x) = x \text{ for } x \in E_0,$$

$$(3) \quad \varphi_t(x) \in \Gamma_j \text{ when } j \in J \text{ and } x \in \Gamma_j,$$

$$(4) \quad \varphi_1 \text{ is Lipschitz.}$$

Note : Γ_0 could be the domain where we want to work.

The closed set E is almost minimal with sliding boundaries Γ_j if

$$(5) \quad \mathcal{H}^d(E \cap B) \leq \mathcal{H}^d(F \cap B) + r^d h(r)$$

when B is a ball of radius r and F is a sliding competitor in B , i.e., $F = \varphi_1(E)$ is as above, with $\varphi_t(x) = x$ for $x \in \mathbb{R}^n \setminus B$ and $\varphi_t(B) \subset B$.

Question: local regularity of these sets, say, with nice L_j .

So far: Ahlfors regularity of E and, under the weird assumption that

(*) each Γ_j either contains E or is at most d -dimensional,
uniform rectifiability of E , stability under limits.

Would be great: - some analogue of monotonicity of density, including for balls centered close to, but not on, the Γ_j ;

- a list of sliding minimal cones, and then

- an approximation result like J. Taylor's;

- getting rid of (*): finding a better proof of Uniform Rectifiability.

Comments :

Seems to be a new question?

Applies to other categories of minimizers.

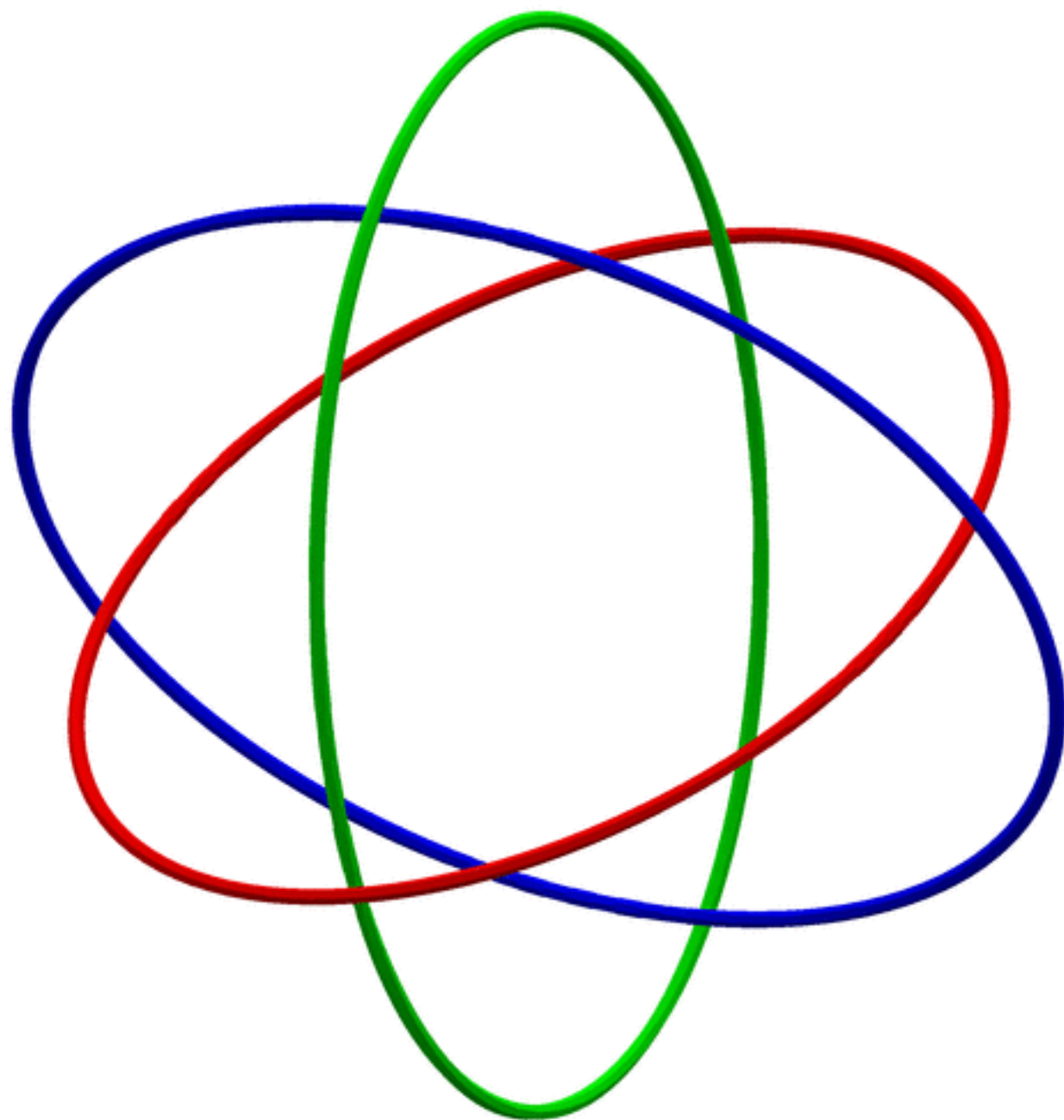
$d = 2$ and $n = 3, 4$ would already be nice.

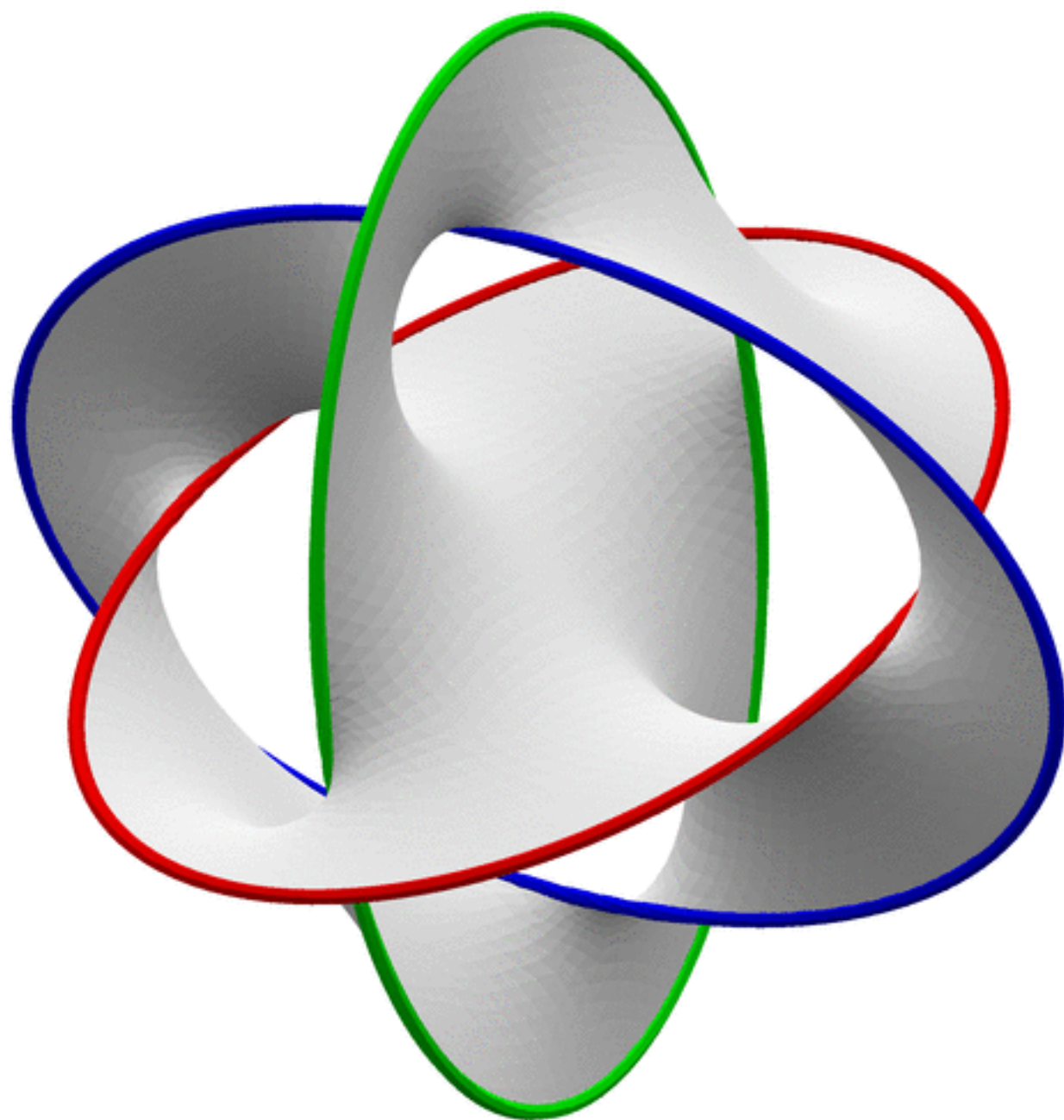
Example : is the cone over the vertices of a cube minimal, with a sliding boundary equal to the great diagonal?

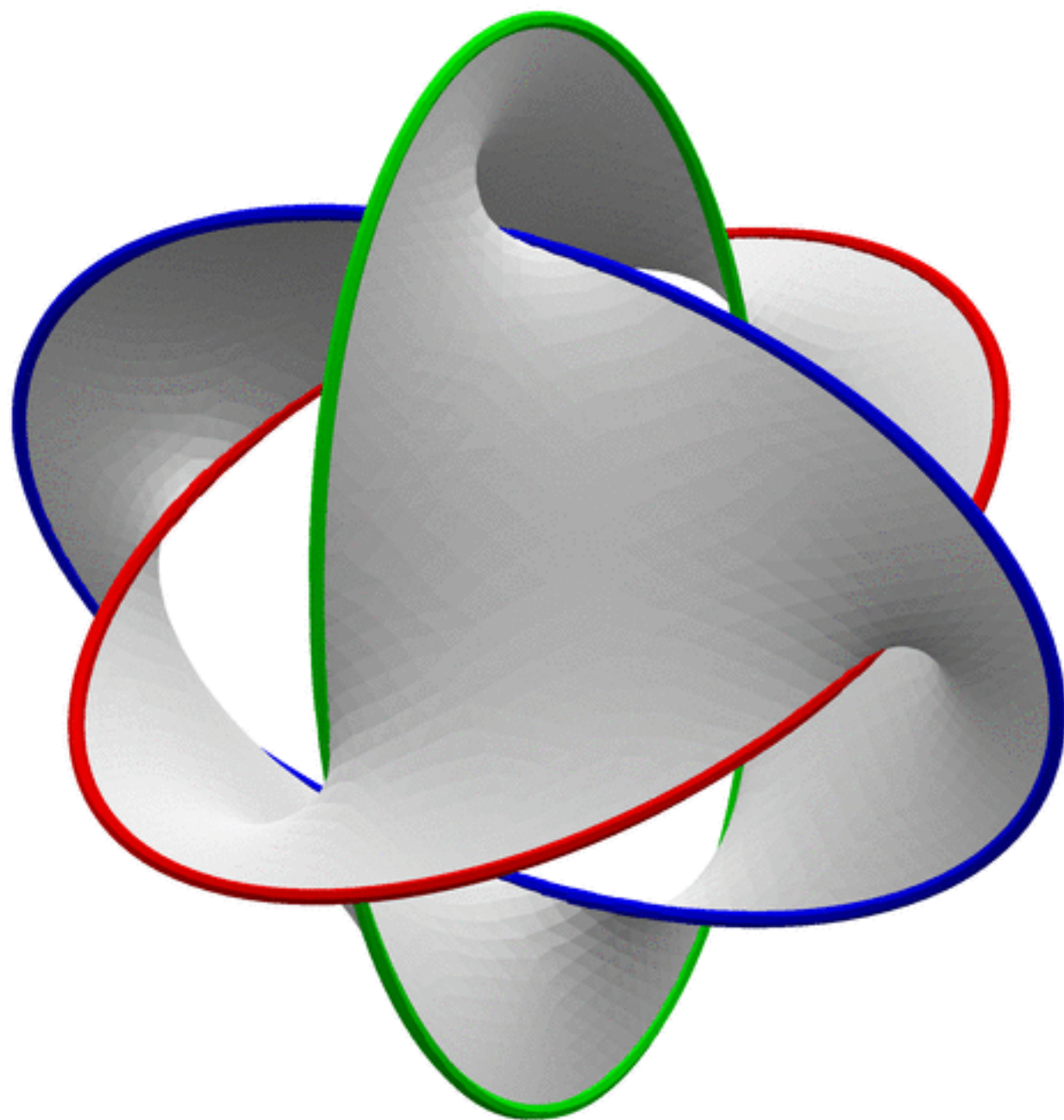
Other topics:

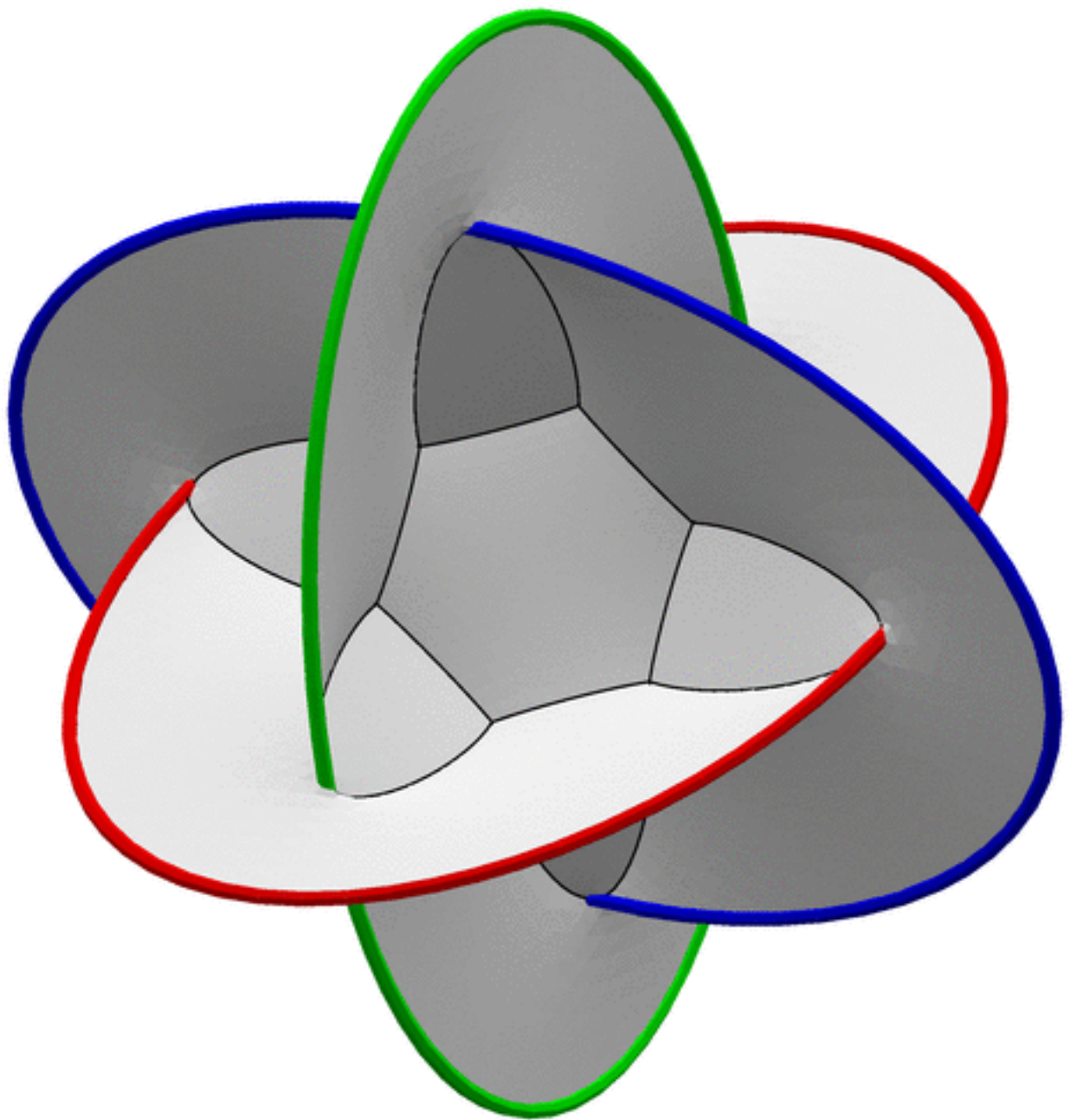
Simpler variants of the Plateau problem:

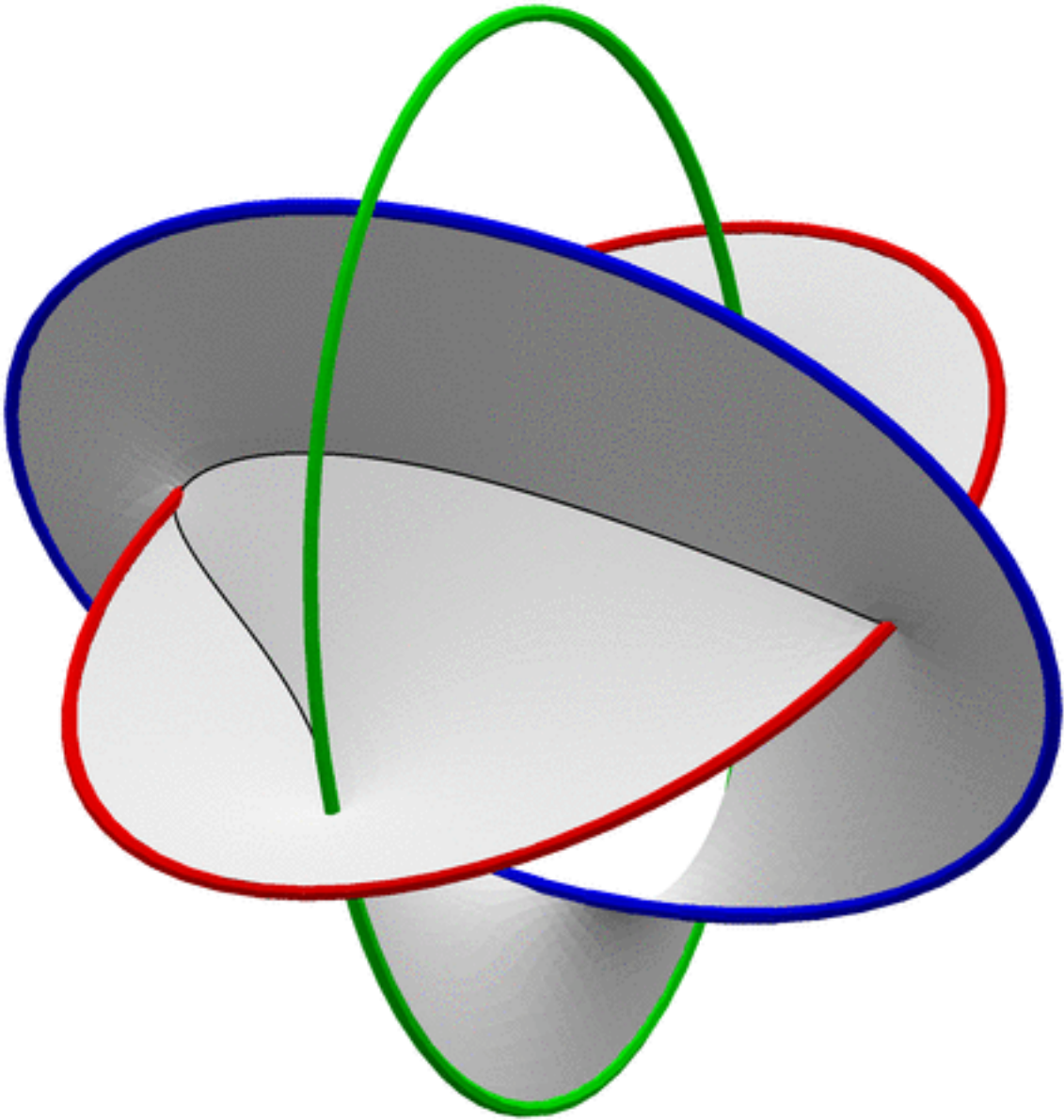
- inside a manifold with complicated topology but no boundary,
- separation conditions,
- homology conditions on the complement.

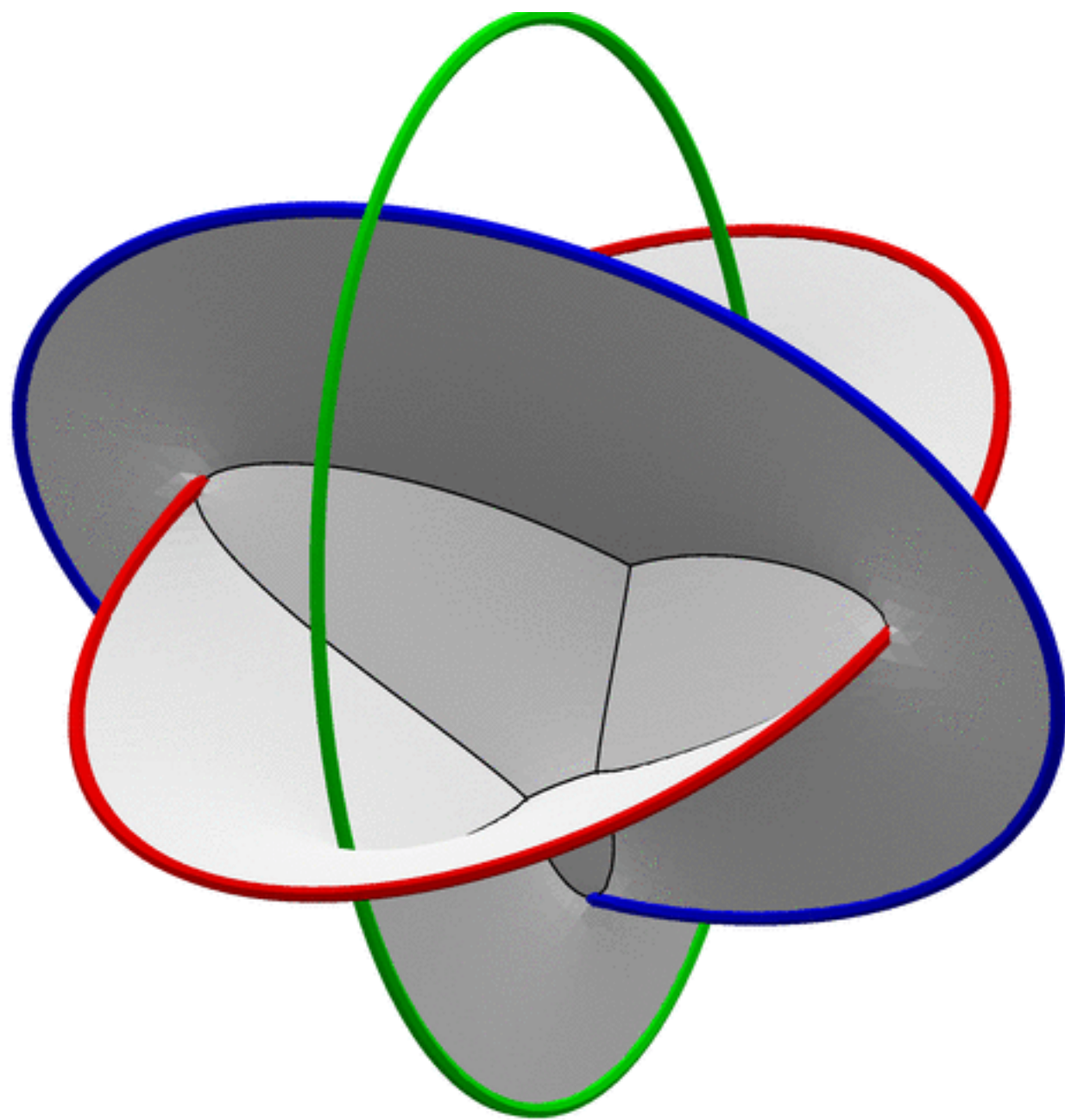












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